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1993 J. Phys. A: Math. Gen. 26 3489

(http://iopscience.iop.org/0305-4470/26/14/016)

J. Phys. A: Math. Gen. **26** (1993) 3489–3501. Printed in the UK

Path integral for coherent states of the dynamical U_2 group and $U_{2|1}$ supergroup

E A Kochetov†

Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, Russia

Received 27 October 1992, in final form 12 March 1993

Abstract. A path-integral formulation in the representation of coherent states for the unitary U_2 group and $U_{2|1}$ supergroup is introduced. U_2 and $U_{2|1}$ path integrals are shown to be defined on the coset spaces $U_2/U_1 \otimes U_1$ and $U_{2|1}/U_{1|1} \otimes U_1$ respectively. These cosets appear as curved classical phase spaces. Partition functions are expressed as path integrals over these spaces. In the case when U_2 and $U_{2|1}$ are the dynamical groups, the corresponding path integrals are evaluated with the help of linear fractional transformations that appear as the group (supergroup) action in the coset space (superspace). Possible applications for quantum models are discussed.

1. Introduction

The standard path integral over fermionic and bosonic variables in the holomorphic representation is widely used in various quantum-mechanical problems. Such an integral can be thought of as an integral over the classical phase space associated with ordinary Fermi and Bose coherent states (CS). These states provide a convenient basis for unitary irreducible representations (UIRs) of the Bose oscillator group and Fermi oscillator supergroup, whose Lie algebras consist of generators

$$\{b^\dagger b, b^\dagger, b, I\} \qquad \text{and} \qquad \{f^\dagger f, f^\dagger, f, I\} \tag{1}$$

respectively. The standard commutation (anticommutation) relations are as follows

$$[b, b^\dagger] = \{f^\dagger, f\} = 1.$$

Bose and Fermi CS can be represented as

$$\begin{aligned} |\alpha\rangle_{\mathrm{B}} &= \exp(-|\alpha|^2/2)\exp(\alpha b^\dagger)|0\rangle_{\mathrm{B}} \\ |\theta\rangle_{\mathrm{F}} &= \exp(\theta\bar{\theta}/2)\exp(\theta f^\dagger)|0\rangle_{\mathrm{F}} \end{aligned} \tag{2}$$

where α is a complex number and θ is a Grassmann parameter. By using decomposition of unity in terms of states (2)

$$\int |\alpha\rangle\langle\alpha||\theta\rangle\langle\theta| \frac{\mathrm{d}\bar{\alpha}\,\mathrm{d}\alpha}{2\pi\mathrm{i}}\,\mathrm{d}\bar{\theta}\,\mathrm{d}\theta = I_{\mathrm{B}} \otimes I_{\mathrm{F}} \equiv I \tag{3}$$

† e-mail:kochetov@theor.jinrc.dubna.su

one can obtain path integral with respect to the measure

$$D\bar{\alpha} D\alpha D\bar{\theta} D\theta \tag{4}$$

which is to be understood as an infinite pointwise product of measures entering into equation (3).

The path measure (4) is invariant under linear shift transformations

$$\alpha \to \alpha + \alpha_0 \qquad \theta \to \theta + \theta_0. \tag{5}$$

The corresponding classical phase space can be thought of as a direct product of a complex plane and a complex flat Grassmann manifold. The Bose (Fermi) oscillator group (supergroup) acts in this space through linear shifts (5). To be more specific, unitary transformations

$$U(g) = \exp(\alpha_0 b^\dagger - \bar{\alpha}_0 b + \theta_0 f^\dagger - f\bar{\theta}_0)$$

induce canonical transformations (5) in the classical phase space.

The flat path integral over measure (4) turns out to be very useful in the framework of perturbation theory, as the unperturbed Hamiltonian H_0 appears to be a linear combination of generators (1). However, it is practically useless in attempts to go beyond the perturbation expansion. This is merely due to the fact that the exact diagonalization of the whole Hamiltonian $H = H_0 + H_{\mathrm{int}}$ requires more general transformations than those of the harmonic-oscillator type.

Let G be a group of transformations that result in diagonalization of H. Then its Lie algebra L is known to contain H as an element. L is then called the spectrum generating algebra (SGA) and the corresponding group G is known as a dynamical group. Note that the direct product of oscillator groups generated by (1) plays the role of the dynamical group for H_0.

The new phase space (an orbit of the coadjoint representation of L) turns out to be a curved one. The path integral over CSs associated with L is to be regarded as an integral in a curved space with a much more complicated measure than that of equation (4). An important point is that G acts in this space via linear fractional transformations, which induce an appropriate change of integration variables in the corresponding path integral.

The concept of CSs associated with the UIRs of Lie group G was first introduced by Perelomov (1972) and generalized to the case of supergroups by Bars and Günaydin (1983). Let us outline below the main features of this approach. Let L be a Lie algebra that has the so-called 3-grading decomposition with respect to the Lie algebra L_0 of its maximal compact subgroup:

$$L = L^{-1} \oplus L^0 \oplus L^{+1} \tag{6}$$

L_0 contains the generator Q of an Abelian U_1 factor, that gives the grading, i.e.

$$L^0 = H \oplus Q$$

and

$$[Q, H] = 0 \qquad [Q, L^{+1}] = L^{+1} \qquad [Q, L^{-1}] = -L^{-1}.$$

The elements $l^m \in L^m$ satisfy the formal commutation relations

$$[l^m, l^n] \in L^{n+n} \qquad n, m = -1, 0, +1 \tag{7}$$

where $L^{n+m} = 0$ for $|n+m| > 1$. For Lie superalgebras the same definitions (6), (7) hold, but the bilinear product (7) is now understood to be either an anticommutator between any two odd elements of superalgebra L or a commutator. The important point concerning the decomposition (6) is that if there exists a set of 'lowest weight' states $|lw\rangle$ that are transformed irreducibly under the maximal compact subgroup action and are annihilated by all the annihilation operators L^{-1}, then the set of states

$$(L^{+1})^p|lw\rangle \qquad p = 0, 1, 2, \ldots \tag{8}$$

form the basis for the irreducible representation of the whole group G. It then follows that the generalized CSs associated with algebra L can be symbolically defined as

$$|CS\rangle = \exp\left(\sum_i l_i^{+1}\alpha_i\right)|lw\rangle \tag{9}$$

where the α_i are even or odd Grassmann parameters (depending on the bosonic or fermionic nature of the raising operators l_i^{+1}). For ordinary Lie groups the α_i are complex numbers. The CS vectors (9) provide a convenient basis for constructing the path-integral representation of systems with dynamical group G, the crucial point being that the irreducibility of the states (8) ensures decomposition of unity in terms of CS (9).

The topological and algebraic structure of the generalized CSs and associated path integrals as well as possible applications to the time-independent and time-dependent systems with dynamical symmetry have been extensively discussed in the review by Zhang *et al* (1990). Note also the paper by Kuratsuji (1981) where the path integral over generalized CSs is expressed as an integral over the complex phase space with a non-trivial geometrical structure.

For physical applications, especially in quantum-optical models, it is convenient to deal with oscillator-like representations of the L-algebra generators. Then, all the L-generators are expressed as bilinears of Bose (Fermi) creation and annihilation operators. As is well known, n^2 bilinears

$$b_i b_j^\dagger \; [b_i, b_j^\dagger] = \delta_{ij} \qquad i, j = 1, 2, \ldots, n$$

generate the Lie algebra of the U_n group. To extend U_n to the unitary $U_{n|m}$ supergroup, one has to add to the n bosonic operators m fermionic ones f_μ, $\mu = 1, 2, \ldots, m$. Bilinears $b_i b_j^\dagger$ and $f_\mu f_\nu^\dagger$ respectively, form the Lie algebras of U_n and U_m under commutation, whereas Bose–Fermi bilinears $b_i f_\mu^\dagger$ and $b_i^\dagger f_\mu$ close into the set $b_i b_j^\dagger$, $f_\mu f_\nu^\dagger$ under anticommutation

$$\{b_i f_\mu^\dagger, b_j^\dagger f_\nu\} = \delta_{ij} f_\mu^\dagger f_\nu + \delta_{\mu\nu} b_j^\dagger b_i. \tag{10}$$

In the subsequent sections we will be concerned with the simplest cases $n = 2$ and $n = 2$, $m = 1$.

2. U_2 Lie algebra in the oscillator-like representation and U_2 CSs

In the Bose-oscillator-like representation the generators of the U_2 Lie algebra can be taken to be

$$K_1 = b_1^\dagger b_1 \qquad K_2 = b_2^\dagger b_2 \qquad K_+ = b_2^\dagger b_1 \qquad K_- = b_1^\dagger b_2. \tag{11}$$

U_2 linear Casimir operator is a number operator $N = b_2^+ b_2 + b_1^+ b_1$. All the higher U_2 Casimirs are functions of N due to the fact that in realization (11) we deal with fully symmetric U_2 representations that are labelled by the eigenvalues of N. As is well known $U_2 = SU_2 \otimes U_1$ which means that the U_2 algebra can be decomposed into the direct sum

$$\{K_+, K_-, K_0 = \tfrac{1}{2}(b_2^+ b_2 - b_1^+ b_1)\} \oplus \{N\} \tag{12}$$

where generators K_+, K_-, K_0 span the SU_2 subalgebra

$$[K_+, K_-] = 2K_0 \qquad [K_0, K_\pm] = \pm K_\pm. \tag{13}$$

Algebra (11) is easily seen to have 3-grading decomposition with respect to the $U_1 \oplus U_1$ subalgebra generated by K_1, K_2:

$$L_+ = \{K_+\} \qquad L_- = \{K_-\} \qquad L_0 = \{K_1, K_2\}$$

grading being achieved with the generator K_2. The lowest-weight state which is transformed irreducibly under $U_1 \otimes U_1$ group action and is annihilated by the K_- operator looks as follows

$$|lw\rangle = |n, 0\rangle \tag{14}$$

where

$$b_1^+ b_1 |n, m\rangle = n|n, m\rangle \qquad b_2^+ b_2 |n, m\rangle = m|n, m\rangle \qquad n, m = 0, 1, 2, \ldots.$$

Due to equation (9) the U_2 CS can be written in the form

$$|\alpha; n\rangle = (1 + |\alpha|^2)^{-n/2} \exp(\alpha b_2^+ b_1)|n, 0\rangle \tag{15}$$

where the complex number α belongs to the coset space $U_2/U_1 \otimes U_1$ which is isomorphic to the complex projective space CP^1. Note that CS (15) depends upon the representation index $n \geqslant 0$—the eigenvalue of the linear Casimir operator. For every value of n the basis in the U_2 representation space can be chosen as

$$|e_p\rangle = |n - p, p\rangle \qquad p = 0, \ldots, n$$

so that $\dim\{|e_p\rangle\} = n + 1$. The overlap of two states $|\alpha'; n\rangle$ and $|\alpha; n\rangle$ is given as

$$\langle \alpha'; n \mid \alpha; n\rangle = (1 + |\alpha'|^2)^{-n/2}(1 + |\alpha|^2)^{-n/2}(1 + \bar{\alpha}'\alpha)^n. \tag{16}$$

An important property of these states is that they satisfy the completeness relation

$$\int |\alpha; n\rangle\langle\alpha; n| \, d\mu_n(\alpha) = I_n = \sum_{p=0}^{n} |e_p\rangle\langle e_p| \tag{17}$$

where the U_2-invariant integration measure looks as follows

$$d\mu_n(\alpha) = \frac{n+1}{\pi}\frac{d^2\alpha}{(1+|\alpha|^2)^2}. \tag{18}$$

It is easily seen that

$$\int \langle\alpha; n|e_p\rangle\langle e_q|\alpha; n\rangle\, d\mu_n(\alpha) = \delta_{pq}. \tag{19}$$

As a result of equation (19), for any operator F acting in the $(n+1)D$ space spanned by $|e_p\rangle$ one has

$$\mathrm{Sp}F = \sum_{p,q}\langle e_p|F|e_q\rangle\delta_{pq} = \int d\mu_n(\alpha)\langle\alpha; n|F|\alpha; n\rangle. \tag{20}$$

The averages over U_2 css look as follows

$$\begin{aligned}
&\langle\alpha; n|K_1|\alpha; n\rangle = \frac{n}{1+|\alpha|^2} \qquad \langle\alpha; n|K_2|\alpha; n\rangle = n\frac{|\alpha|^2}{1+|\alpha|^2}\\
&\langle\alpha; n|(K_+)^p|\alpha; n\rangle = \frac{n!}{(n-p)!}\frac{\bar{\alpha}^p}{(1+|\alpha|^2)^p}\\
&\langle\alpha; n|(K_-)^p|\alpha; n\rangle = \frac{n!}{(n-p)!}\frac{\alpha^p}{(1+|\alpha|^2)^p}\\
&\langle\alpha; n|K_0|\alpha; n\rangle = \frac{n}{2}\frac{1-|\alpha|^2}{1+|\alpha|^2} \qquad p = 0, 1, \ldots, n.
\end{aligned} \tag{21}$$

3. Path integral

Let us consider the path integral over U_2 css for the partition function

$$Z = \mathrm{Sp}\,e^{-\beta H}$$

where the Hamiltonian H belongs to the U_2 enveloping algebra. From equation (20) one has

$$Z = \sum_{n=0}^{\infty}\int d\mu_n(\alpha)\langle\alpha; n|e^{-\beta H}|\alpha; n\rangle.$$

Defining ϵ as β/N and using equation (17) we write, in the usual manner,

$$\begin{aligned}
Z = \sum_{n=0}^{\infty}\int d\mu_n(\alpha)\langle\alpha; n|\alpha_N; n\rangle\langle\alpha_N; n|e^{-\epsilon H}|\alpha_{N-1}; n\rangle\langle\alpha_{N-1}; n|e^{-\epsilon H}\\
\ldots e^{-\epsilon H}|\alpha_0; n\rangle\langle\alpha_0; n|\alpha; n\rangle\, d\mu_n(\alpha_N)\ldots d\mu_n(\alpha_0).
\end{aligned} \tag{22}$$

Up to the second order in ϵ one has

$$\langle\alpha_j; n|\mathrm{e}^{-\epsilon H}|\alpha_i; n\rangle = \langle\alpha_j; n|\alpha_i; n\rangle \exp(-\epsilon\mathcal{H}_n(\bar{\alpha}_j, \alpha_i))$$

where

$$\mathcal{H}_n(\bar{\alpha}_j, \alpha_i) = \langle\alpha_j; n|H|\alpha_i; n\rangle / \langle\alpha_j; n|\alpha_i; n\rangle.$$

The integration over $\mathrm{d}\mu_n(\alpha)$ in accordance with equation (20) yields

$$\int \mathrm{d}\mu_n(\alpha)\langle\alpha; n|\alpha_N; n\rangle\langle\alpha_0; n|\alpha; n\rangle = \mathrm{Sp}|\alpha_N; n\rangle\langle\alpha_0; n| = \langle\alpha_0; n|\alpha_N; n\rangle$$

so that

$$Z = \sum_{n=0}^{\infty} \lim_{N\to\infty} \int \prod_{i=0}^{N} \mathrm{d}\mu_i \prod_{j=1}^{N} \langle\alpha_j|\alpha_{j-1}\rangle\langle\alpha_0|\alpha_N\rangle \exp\left(-\epsilon \sum_{j=1}^{N} \mathcal{H}_n(\bar{\alpha}_j, \alpha_{j-1}) \right). \tag{23}$$

It can easily be checked that the terms of $\mathcal{O}(\epsilon^2)$, $\epsilon \to 0$ in $\langle\alpha_j; n|\mathrm{e}^{-\epsilon H}|\alpha_i; n\rangle$ do not affect the limiting form (23) (Berezin 1971).

For any state vector $|\psi\rangle$ which belongs to the $(n+1)D$ Hilbert space with an element

$$P_m(\alpha)/(1+|\alpha|^2)^{n/2}$$

where $P_m(\alpha)$ is an arbitrary polynomial of degree $m \leqslant n$ (Perelomov 1972), one has

$$\langle\psi| = \int \mathrm{d}\mu_n(\alpha)\langle\psi|\alpha\rangle\langle\alpha|$$

which can be written in its components as

$$\psi_n(\beta) = \int \mathrm{d}\mu_n(\alpha)\langle\alpha; n|\beta; n\rangle\psi_n(\alpha) \qquad \psi_n(\alpha) \equiv \langle\psi|\alpha; n\rangle. \tag{24}$$

Note that the reproducing kernel $\langle\alpha; n|\beta; n\rangle$ acts as a delta function with respect to the measure $\mathrm{d}\mu_n(\alpha)$. Due to equation (24) the integration over $\mathrm{d}\mu_0$ in (23) can be carried out explicitly to yield

$$Z = \sum_{n=0}^{\infty} \lim_{N\to\infty} \int \mathrm{d}\mu_1 \ldots \mathrm{d}\mu_N \prod_{j=1}^{N} \langle\alpha_j|\alpha_{j-1}\rangle \exp\left(-\epsilon \sum_{j=1}^{N} \mathcal{H}_n(\bar{\alpha}_j, \alpha_{j-1}) \right)_{\alpha_0=\alpha_N}. \tag{25}$$

With $\alpha_{j-1} = \alpha_j - \delta_j$ it then follows

$$\ln\langle\alpha_j|\alpha_{j-1}\rangle = \frac{n}{2}\frac{\alpha_j\bar{\delta}_j - \bar{\alpha}_j\delta_j}{1+|\alpha_j|^2} + \mathrm{O}(\delta_j^2). \tag{26}$$

In the continuous limit this yields

$$Z = \sum_{n=0}^{\infty} \int_{\alpha(0)=\alpha(\beta)} D\mu_n(\alpha) \exp\left(-\frac{n}{2}\int_0^{\beta} \frac{\bar{\alpha}\dot{\alpha} - \dot{\bar{\alpha}}\alpha}{1+|\alpha|^2}\,\mathrm{d}s - \int_0^{\beta} \mathcal{H}_n(\bar{\alpha}, \alpha)\,\mathrm{d}s \right) \tag{27}$$

where the following normalization holds:

$$\int_{\alpha(0)=\alpha(\beta)} D\mu_n(\alpha) \exp\left(-\frac{n}{2}\int_0^{\beta} \frac{\bar{\alpha}\dot{\alpha} - \dot{\bar{\alpha}}\alpha}{1+|\alpha|^2}\,\mathrm{d}s \right) = \mathrm{Sp}\, I_n = n+1.$$

Let us note that the partition function Z could be evaluated directly through the pointwise representation (22), provided an explicit transformation of the CSs under an appropriate group action to first order in ϵ is known. This has recently been done for $\mathrm{SU}_{1,1}$ dynamical systems by Gerry (1989) and later for the $\mathrm{SL}_{2;C}$ case by Ellinas (1992). In this way, one is led to a set of recursion relations on the group parameters which, in the limit $\epsilon \to 0$, gives rise to a set of coupled first-order differential equations. The whole procedure requires long calculations even in the simplest cases. In the next section, we will show that there exists a direct way to evaluate the sum over paths in equation (27).

4. Evaluation of the U_2 path integral

In order to illustrate how the general formula (27) works, let us take H in the form

$$H = \omega_1 b_1^\dagger b_1 + \omega_2 b_2^\dagger b_2 + \bar{\lambda} b_2^\dagger b_1 + \lambda b_2 b_1^\dagger \tag{28}$$

where $\omega_1\omega_2 \geqslant |\lambda|^2$ for the Hamiltonian (28) is to be bounded from below. For the partition function in accordance with equations (27) and (21) one gets

$$Z = \sum_{n=0}^{\infty} e^{-n\beta(\omega_1+\omega_2)/2} \int_{\alpha(0)=\alpha(\beta)} D\mu_n(\alpha) \exp\left(-\int_0^\beta \mathcal{L}_n(\bar{\alpha},\alpha)\,ds\right) \tag{29}$$

where the quantity

$$\mathcal{L}_n = \frac{n}{2}\frac{\bar{\alpha}\dot{\alpha} - \dot{\bar{\alpha}}\alpha}{1+|\alpha|^2} + \frac{n}{2}(\omega_1-\omega_2)\frac{1-|\alpha|^2}{1+|\alpha|^2} + n\bar{\lambda}\frac{\bar{\alpha}}{1+|\alpha|^2} + n\lambda\frac{\alpha}{1+|\alpha|^2} \tag{30}$$

can be defined as the Lagrangian. The Euler–Lagrange equations lead to the equations of motion

$$\dot{\alpha} = \{\mathcal{H}_n, \alpha\}$$

where $\{,\}$ is a Poisson bracket defined by

$$\{A, B\} = \frac{(1+|\alpha|^2)^2}{n}\left[\frac{\partial A}{\partial \alpha}\frac{\partial B}{\partial \bar{\alpha}} - \frac{\partial A}{\partial \bar{\alpha}}\frac{\partial B}{\partial \alpha}\right].$$

This indicates that the classical phase space spanned by α and $\bar{\alpha}$ is curved—in fact the complex projective line $CP^1 \simeq S^2$ (Berezin 1975).

Due to the relation $U_2 = SU_2 \otimes U_1$ the general U_2 transformation can be taken to be

$$U_2 = \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} e^{i\phi}$$

where $|u|^2 + |v|^2 = 1$ and $0 \leqslant \phi < 2\pi$. The path integral in equation (27) can be evaluated with the help of transformations of the integration variables which are induced by the U_2 action in the coset space $U_2/U_1 \otimes U_1$. U_2 acts in the integration space $U_2/U_1 \otimes U_1$ through the following canonical transformations:

$$\alpha \to \alpha = (u\alpha + v)/(-\bar{v}\alpha + \bar{u}) \tag{31}$$

where the group parameters u and v are kept constant. The integration measure $d\mu_n(\alpha)$ is invariant under transformations (31). The same is true for the kinetic term, for example

$$\begin{aligned}\int_0^\beta \frac{\bar{\alpha}\dot{\alpha}}{1+|\alpha|^2}\,ds &\to \int_0^\beta \frac{\dot{\alpha}}{1+|\alpha|^2}\frac{\bar{u}\bar{\alpha}+\bar{v}}{-\bar{v}\alpha+\bar{u}}\,ds \\ &= \int_0^\beta \frac{\bar{\alpha}\dot{\alpha}}{1+|\alpha|^2}\,ds + \int_0^\beta \frac{ds}{1+|\alpha|^2}\left(\dot{\alpha}\frac{\bar{u}\bar{\alpha}+\bar{v}}{-\bar{v}\alpha+\bar{u}} - \bar{\alpha}\dot{\alpha}\right) \\ &= \int_0^\beta \frac{\bar{\alpha}\dot{\alpha}}{1+|\alpha|^2}\,ds - \int_0^\beta d\ln(\bar{v}\alpha - \bar{u}) = \int_0^\beta \frac{\bar{\alpha}\dot{\alpha}}{1+|\alpha|^2}\,ds\end{aligned}$$

where the total derivative can be dropped since $\alpha(0)=\alpha(\beta)$.

Upon taking

$$
\begin{aligned}
&u=\frac{\omega}{|\omega|}\sqrt{\frac{\bar\lambda}{|\lambda|}}\cos\theta \qquad v=\sqrt{\frac{\bar\lambda}{|\lambda|}}\sin\theta \\
&\cos\theta=\frac{1}{\sqrt{2}}(1+|\omega|/\Omega)^{1/2} \qquad \sin\theta=\frac{1}{\sqrt{2}}(1-|\omega|/\Omega)^{1/2} \\
&\Omega=\sqrt{\omega^2+|\lambda|^2} \qquad \omega=\tfrac{1}{2}(\omega_1-\omega_2)
\end{aligned}
\tag{32}
$$

one eliminates the terms linear in λ and $\bar\lambda$ in the exponent of equation (29) which results in

$$
Z=\sum_n^\infty \exp\left(-n\beta\frac{\omega_1+\omega_2}{2}\right)Z_n
$$

where the path integral

$$
Z_n=\int_{\alpha(0)=\alpha(\beta)} D\mu_n(\alpha)\exp\left(-\frac{n}{2}\int_0^\beta\frac{\bar\alpha\dot\alpha-\dot{\bar\alpha}\alpha}{1+|\alpha|^2}\,ds-n\Omega\int_0^\beta\frac{1-|\alpha|^2}{1+|\alpha|^2}\,ds\right)
\tag{33}
$$

can be evaluated directly through the definition (25):

$$
Z_n=\lim_{N\to\infty}\int_{\alpha_0=\alpha_N} d\mu_1\ldots d\mu_N\prod_{j=1}^N\langle\alpha_j|e^{2\epsilon\Omega K_0}|\alpha_{j-1}\rangle
$$

i.e. by taking into account that

$$
e^{2\epsilon\Omega K_0}|\alpha_{j-1}\rangle=e^{-\epsilon\Omega n}|\alpha_{j-1}e^{2\epsilon\Omega}\rangle
$$

one gets

$$
Z_n=e^{-\beta\Omega n}\lim_{N\to\infty}\int_{\alpha_0=\alpha_N} d\mu_1\ldots d\mu_N\prod_{j=1}^N\langle\alpha_j|\alpha_{j-1}e^{2\epsilon\Omega}\rangle.
\tag{34}
$$

By using decomposition of unity (17) the integration over $d\mu_1\ldots d\mu_{N-1}$ can be carried out explicitly. This yields

$$
\begin{aligned}
Z_n e^{\beta\Omega n}&=\int_{\alpha_N=\alpha_0} d\mu_N\langle\alpha_N|\alpha_0 e^{2N\epsilon\Omega}\rangle=\int d\mu(\alpha)\langle\alpha|\alpha e^{2\beta\Omega}\rangle \\
&=(n+1)e^{2\beta\Omega(n+1)}\int_0^\infty dx\frac{(1+x)^n}{(e^{2\beta\Omega}+x)^{n+2}}=\frac{e^{2\beta\Omega(n+1)}-1}{e^{2\beta\Omega}-1}
\end{aligned}
$$

which gives

$$
Z=\sum_{n=0}^\infty\exp\left(-n\beta\frac{\omega_1+\omega_2}{2}\right)\frac{\sinh\beta\Omega(n+1)}{\sinh\beta\Omega}.
$$

It then follows that

$$Z = \sum_{n=0}^{\infty} \exp\left(-n\beta \frac{\omega_1 + \omega_2}{2}\right) \sum_{m=-n/2}^{m=n/2} \exp(-2\beta\Omega m)$$

which yields the correct spectrum

$$E_{n_1,n_2} = \tfrac{1}{2}(\omega_1 + \omega_2)(n_1 + n_2) + \Omega(n_1 - n_2) \qquad n_1, n_2 \geqslant 0.$$

At the end of this section it should be pointed out that the CSs (15), in fact, coincide with those of the SU_2 group which can be parametrized by the points of the coset space SU_2/U_1 (Perelomov 1972). This is merely due to the fact that $U_2 = SU_2 \otimes U_1$. Thus, in order to construct a path integral for a spin system with dimensionality $2j+1$ one can employ equations (21) where one must put $j = \frac{1}{2}n$. For example, for the linear spin Hamiltonian

$$H = \Omega K_0 + \bar{\lambda} K_+ + \lambda K_- \qquad K_0|m\rangle = m|m\rangle \qquad m = -j, -j+1, \ldots, j$$

one gets

$$\begin{aligned} Z_j = \mathrm{Sp}\, e^{-\beta H_j} = \int D\mu_j(\alpha) \exp\bigg(&- j \int_0^\beta \frac{\bar{\alpha}\dot{\alpha} - \dot{\bar{\alpha}}\alpha}{1+|\alpha|^2}\, \mathrm{d}s + \Omega j \int_0^\beta \frac{1-|\alpha|^2}{1+|\alpha|^2}\, \mathrm{d}s \\ &- 2j\bar{\lambda} \int_0^\beta \frac{\bar{\alpha}}{1+|\alpha|^2}\, \mathrm{d}s - 2j\lambda \int_0^\beta \frac{\alpha}{1+|\alpha|^2}\, \mathrm{d}s \bigg) \end{aligned} \qquad (35)$$

which is readily evaluated with the help of substitution (31).

5. Path integral over $U_{2|1}$ CS

Thus far, we have discussed the path-integral representations for the U_2 Lie algebra. From the physical point of view ordinary Lie algebras are relevant for purely bosonic (or fermionic) systems. For example, the partition function for a superfluid helium model is expressed as a path integral over CSs associated with the non-compact $SU_{1,1}$ algebra (Gerry and Silverman 1982). In the models of quantum optics, however, Hamiltonians that include both bosonic and fermionic degrees of freedom principally appear. One needs then to consider path integrals over super CSs associated with underlying superalgebras. For example, the compact $U_{1|1}$ and non-compact $OSP_{2|2}$ superalgebras turn out to be SGAs for Jaynes–Cummings and Rabi Hamiltonians, respectively (Buzano *et al* 1989). Note also that it has recently been considered the path integral for $OSP_{1|2}$ CSs (Schmitt and Mufti 1991). Here we consider the simplest 9D unitary $U_{2|1}$ supergroup that appears as the dynamical group for various quantum-optical Hamiltonians.

In the oscillator-like representation the $U_{2|1}$ generators can be taken to be (Bars and Günadyin 1983)

$$\begin{aligned} &L_0 = \{b_1^\dagger b_1, f^\dagger f, b_1^\dagger f, b_1 f^\dagger\} \oplus \{b_2^\dagger b_2\} \\ &L_+ = \{b_2^\dagger f, b_2^\dagger b_1\} \qquad L_- = \{b_2 f^\dagger, b_2 b_1^\dagger\} \end{aligned} \qquad (36)$$

where $\{f, f^\dagger\} = 1$. It then follows that equation (36) gives the 3-grading decomposition with respect to the maximal compact subsupergroup with superalgebra $U_{1|1} \oplus U_1$. Grading is achieved with the operator $b_2^\dagger b_2$. The operators in the first curly brackets in L_0 form the basis for the $\mathrm{U}_{1|1}$ superalgebra. As is known, all the irreps. of $\mathrm{U}_{1|1} \otimes \mathrm{U}_1$ superalgebra are 1D or 2D (de Crombrugghe and Rittenberg 1983). The basis can be taken as

$$\text{1D case: } |e_0\rangle = |0, m\rangle_\mathrm{B} \otimes |0\rangle_\mathrm{F}$$
$$\text{2D case: } |e_1\rangle = |n-1, m\rangle_\mathrm{B} \otimes |1\rangle_\mathrm{F} \qquad |e_2\rangle = |n, m\rangle_\mathrm{B} \otimes |0\rangle_\mathrm{F}$$

where use is made of the standard notation so that

$$f|0\rangle_\mathrm{F} = 0 \qquad f^\dagger|0\rangle_\mathrm{F} = |1\rangle_\mathrm{F}.$$

The $\mathrm{U}_{2|1}$ supergroup acts in the superspace which is formed as the Grassmann envelope of the $\mathrm{U}_{2|1}$ superalgebra representation space (see, for example, Berezin and Tolstoy 1981). The basis of this superspace is given as $|e_0\rangle$ in the 1D case and $|e_2\rangle, \zeta|e_1\rangle$ in the 2D case. Here ζ is a Grassmann parameter and vector $|e_1\rangle$ is chosen to have an odd grading (consequently, $|e_2\rangle$ is even graded).

The lowest-weight vectors that are transformed irreducibly under the $\mathrm{U}_{1|1} \otimes \mathrm{U}_1$ supergroup action and are annihilated by all L_- operators are as follows

$$|0,0\rangle_\mathrm{B} \otimes |0\rangle_\mathrm{F} \qquad \text{and} \qquad |n,0\rangle_\mathrm{B} \otimes |0\rangle_\mathrm{F} + \zeta|n-1,0\rangle_\mathrm{B} \otimes |1\rangle_\mathrm{F}. \tag{37}$$

Due to formula (9) the $\mathrm{U}_{2|1}$ CS can be represented in the form

$$|\alpha, \theta; n\rangle = (1+|\alpha|^2)^{-n/2} \exp\left(-\frac{n}{2}\frac{\bar\theta\theta}{1+|\alpha|^2}\right) \exp(\alpha b_2^\dagger b_1)(|n,0\rangle_\mathrm{B}|0\rangle_\mathrm{F} + \sqrt{n}\theta|n-1,0\rangle_\mathrm{B}|1\rangle_\mathrm{F}). \tag{38}$$

Notice that CS (38) depends upon ordinary and Grassmann parameters simultaneously while the representation index n is now an eigenvalue of the $\mathrm{U}_{2|1}$ linear Casimir operator

$$N = b_1^\dagger b_1 + b_2^\dagger b_2 + f^\dagger f.$$

The overlap of the two states (38) is

$$\langle\alpha', \theta'; n|\alpha, \theta; n\rangle = \langle\alpha'; n|\alpha; n\rangle \exp\left(-\frac{n}{2}\frac{\bar\theta\theta}{1+|\alpha|^2} - \frac{n}{2}\frac{\bar\theta'\theta'}{1+|\alpha'|^2} + n\frac{\bar\theta'\theta}{1+\bar\alpha'\alpha}\right) \tag{39}$$

where $\langle\alpha'; n|\alpha; n\rangle$ is given by equation (16). Unity in the representation n is resolved as

$$I_n = \int |\alpha, \theta; n\rangle\langle\alpha, \theta; n|\, \mathrm{d}\mu_n(\alpha, \theta) \tag{40}$$

where the $\mathrm{U}_{2|1}$ invariant measure reads as follows

$$\mathrm{d}\mu_n = \exp\left(\frac{\theta\bar\theta}{1+|\alpha|^2}\right) \frac{\mathrm{d}^2\alpha}{1+|\alpha|^2} \frac{\mathrm{d}\bar\theta\mathrm{d}\theta}{\pi}. \tag{41}$$

The $U_{2|1}$ path integral can be obtained by the same procedure as that used in the U_2 case. The new point is that a more complicated kinetic term appears and the antiperiodic boundary conditions for θ are:

$$Z = \mathrm{Sp}\,\mathrm{e}^{-\beta H} = \sum_n \int_{\alpha(0)=\alpha(\beta),\theta(0)=-\theta(\beta)} D\mu_n(\alpha,\theta) \exp\left(-\frac{n}{2}\int_0^\beta \frac{\dot{\bar{\alpha}}\alpha - \bar{\alpha}\dot{\alpha}}{1+|\alpha|^2}\,\mathrm{d}s \right.$$
$$\left. + \frac{n}{2}\int_0^\beta \frac{\bar{\theta}\dot{\theta} - \dot{\bar{\theta}}\theta}{1+|\alpha|^2}\,\mathrm{d}s + \frac{n}{2}\int_0^\beta \frac{(\dot{\bar{\alpha}}\alpha - \bar{\alpha}\dot{\alpha})\bar{\theta}\theta}{(1+|\alpha|^2)^2}\,\mathrm{d}s - \int_0^\beta H(\alpha,\theta)\,\mathrm{d}s \right) \qquad (42)$$

where

$$H(\alpha,\theta) = \langle \alpha,\theta; n|H|\alpha,\theta; n\rangle.$$

In the case when $U_{2|1}$ is a spectrum generating algebra, i.e. H belongs to the $U_{2|1}$ even subalgebra, path integral (42) can be calculated by a change in integration variables in accordance with the $U_{2|1}$ group action in the classical phase space which is isomorphic to the coset $U_{2|1}/U_{1|1} \otimes U_1$.

Namely, the $U_{2|1}$ supergroup element in the fundamental representation can be defined as

$$U = \begin{pmatrix} \omega_1 & \theta_1 & \lambda_1 \\ \theta_2 & \omega_2 & \theta_3 \\ \lambda_2 & \theta_4 & \omega_3 \end{pmatrix} \qquad U^\dagger U = 1$$

where $\omega_{1,2,3}$ and $\lambda_{1,2}$ are even Grassmann parameters and $\theta_{1,2,3,4}$ are the odd ones. Then, under the $U_{2|1}$ action the supervariable $(\alpha,\theta) \in U_{2|1}/U_{1|1} \otimes U_1$ undergoes a linear fractional transformation (Bars and Günaydin 1983)

$$\alpha \to \frac{\omega_1\alpha + \theta_1\theta + \lambda_1}{\lambda_2\alpha + \theta_4\theta + \omega_3} \qquad \theta \to \frac{\theta_2\alpha + \omega_2\theta + \theta_3}{\lambda_2\alpha + \theta_4\theta + \omega_3}. \qquad (43)$$

The integration measure in equation (42) remains invariant under transformations (43). By specifying the $U_{2|1}$ parameters in equation (43) in the same manner as in the previous section for the U_2 case, one can evaluate the path integral (42). This is equivalent to the direct diagonalization of the Hamiltonian by means of appropriate $U_{2|1}$ rotation in the super Fock space.

6. Conclusions

In conclusion, some remarks have to be made. First of all, it should be pointed out that we are dealing with CSs associated with finite-dimensional Lie algebras (superalgebras). As a consequence, only quantum systems with finite degrees of freedom are being considered.

The next point is that the path integral over U_2 and $U_{2|1}$ CSs turns out to be very convenient in the semiclassical treatment. In the general quantization scheme for curved phase spaces developed by Berezin (1975) the representation index n labelling CSs associated with a group of motions of this phase space plays the role of $1/\hbar$. In the oscillator-like representation this means a large particle number limit. The stationary-phase method for integrals (27) and (42) as $n \to \infty$ leads to the classical Euler–Lagrange equations. The important point is that by means of the substitution $\alpha \to \alpha/\sqrt{n}$ the integral (27) in the limit

$n \to \infty$ goes to the 'flat' one with the standard measure $D\alpha D\bar{\alpha}$. Thus, one can employ standard methods in dealing with the SU_2 path integral in the limit of a large total spin. This is in complete accordance with the fact that SU_2 CSs at large values of spin j go over into the ordinary (Glauber) CSs (Perelomov 1972). For example, the partition function (35) in the limit $j \to \infty$ reads as

$$Z_j = e^{\Omega\beta j} \int_{\alpha(0)=\alpha(\beta)} D\alpha D\bar{\alpha} \exp\left(\frac{1}{2}\int_0^\beta (\dot{\bar{\alpha}}\alpha - \bar{\alpha}\dot{\alpha})\,ds - \Omega\int_0^\beta |\alpha|^2\,ds - \sqrt{2j}\right.$$
$$\left. \times \int_0^\beta \bar{\lambda}\bar{\alpha}\,ds - \sqrt{2j}\int_0^\beta \lambda\alpha\,ds\right) \tag{44}$$

where the coefficient λ is supposed to be time-dependent. The path integral (44) is seen to be easily calculated (see, for example, Dacol 1980).

As another example the nuclear Hamiltonian proposed by Lipokin, Meshkov and Glik (LMG) which in the spin representation reads as (Lipkin *et al* 1965)

$$H = \epsilon\left[K_0 + \frac{r}{4j}(K_+^2 + K_-^2)\right]$$

where the K_i are the SU_2 generators of dimensionality $2j+1$ with $j = \frac{1}{2}n$ could be considered. Here n is the total number of particles in the LMG model, ϵ and r are real parameters. Note that H belongs to the SU_2 enveloping algebra. With the help of equations (21) at $p = 2$ one can readily obtain the path-integral representation for the LMG model in the form of equation (27). It should be noted that a Bohr–Sommerfeld quantization procedure based on the SU(2) path-integral representation with large $j = n/2$ has successfully been applied to the LMG model by Shankar (1980).

It should also be pointed out that representations (27) and (42) hold for Hamiltonians that belong to the U_2 and $U_{2|1}$ enveloping algebras, as has just been mentioned for the SU_2 case.

The final remark concerns linear fractional transformations (31) and (43). The U_2 and $U_{2|1}$ integration measures remain invariant under the corresponding local linear fractional transformations. This means that one could try to use them in calculating U_2 and $U_{2|1}$ path integrals with parameters depending on time. These and related problems will be discussed elsewhere.

Acknowledgment

This work was supported by the 'Hubbard' project 91112 of the Russian state programme on high-temperature superconductivity.

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